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On the Macroscopic Description of Slow Viscous Flow Past a Random Array of Spheres

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We give rigorous derivation of Brinkman's equation as the effective equation for slow viscous flow in porous media with high porosity. The medium is composed of spherical obstacles distributed randomly, and the microscopic flow is described by the Stokes equation. Along the road we give W_2^1 convergence estimates for the point-sources approximation.

KEY WORDS: Random media, fluid mechanics.

1. INTRODUCTION

The derivation of macroscopic equations for flow in porous media is a long-standing problem of highly practical and theoretical importance. One of the equations commonly used is Brinkman's equation (Brinkman,[2] Saffman[15]). In this paper we show that the solution to this equation is indeed the limit (in the appropriate sense) of the flow of slow viscous incompressible fluid through a random distribution of spheres (which is our model for a porous medium).

We assume that the microscopic flow is described by Stokes equation (Happel and Brenner[7]), and that the number of spheres N goes to ∞ as their radius goes to zero under the scaling

$$R = \frac{\alpha'}{N}, \qquad \alpha' = 0(1) \tag{1.1}$$

In the last decade there several works appeared which analyzed boundary value problems for the Laplace equation and the heat equation in a

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similar setup (Kac,[9] Rauch and Taylor,[14] Hruslov and Marchenko,[8] Papanicolaou and Varadhan,[13] Ozawa,[12] Figari, Orlandi, and Teta[5]). Our approach is similar to the one suggested first by Ozawa[12] and then extended and clarified by Figari, Orlandi, and Teta.[5] The idea is first to approximate the full microscopic problem by point sources and then to average those singularities in order to obtain the desired smooth field. There is, however, a major difference between our proof and the works we mentioned above: Stokes equation does not have a maximum principle, nor does it have an obvious probabilistic interpretation, so that the arguments used for the elliptic problems do not go through here. Instead, we use in Section 3 variational principles to establish the convergence of the point-sources approximation to the microscopic flow. Interestingly, we obtain estimates in the W_2^1 norm in contrast to the L_p estimates obtained in Refs. 12 and 5.

In Section 2 we formulate the physical problem, describe the underlying probability distributions, and give a few preliminary lemmas. The point sources are averaged out in Section 4 where we show that they converge to the smooth solution of Brinkman's equation. This part is along the same line as Figari, Orlandi, and Teta[5] and actually can be abstracted so that it holds for a large family of operators. The reader can consult Brinkman,[2] Childress,[4] Caflisch and Rubinstein,[3] Lundgren,[11] Saffman,[15] or Tam[16] (among others) for formal derivations and various applications of the Brinkman's equation.

2. FORMULATION AND BASIC ESTIMATES

Let $D \subseteq R^3$ be an open smooth domain. $\{\mathbf{y}_j\}$ $j = 1,..., N$ are a collection of points in D that are independent identically distributed random variables with density function $\rho(\mathbf{x})$ which is continuous with compact support in D. There is a sphere B_j^N centered at each y_j with radius

$$R = \frac{\alpha'}{N}, \qquad \alpha' = 0(1) \tag{2.1}$$

Set $D_1^N = D - \bigcup_{j=1}^N B_j^N$, and consider in D_1^N the problem

$$\begin{aligned} \mu\, \Delta \mathbf{u}^N - \lambda \mathbf{u}^N &= \nabla P^N + \mathbf{f} \qquad \text{in} \quad D_1^N \\ \nabla \cdot \mathbf{u}^N &= 0 \qquad \text{in} \quad D_1^N \\ \mathbf{u}^N = 0 \quad \text{on} \quad S_j^N & \qquad j = 1, 2,..., N \end{aligned} \tag{2.2}$$

Here μ is the viscosity, $\mathbf{u}$ the velocity, and P the pressure, while S_j^N is the surface of B_j^N and $\mathbf{f}(\mathbf{x})$ is a continuous vector field with compact support in D. (We concentrate here on $D = R^3$, so we do not prescribe boundary conditions on ∂D. It is easy to extend our proofs to bounded domains).

Equation (2.2) is the Laplace transform of the time-dependent Stokes equation. The fundamental tensor for that equation in free space is given by $\boldsymbol{\psi}$ where

$$(\boldsymbol{\psi})_{ij} = \frac{1}{4\pi\mu}\left[\frac{e^{-\sigma|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}\delta_{ij} + \frac{\partial^2}{\partial x_i\,\partial x_j}\left(\frac{1-e^{-\sigma|\mathbf{x}-\mathbf{y}|}}{\sigma^2|\mathbf{x}-\mathbf{y}|}\right)\right] \tag{2.3}$$

so that

$$\mathbf{w} = \boldsymbol{\psi}\mathbf{e}$$

solves

$$\mu\,\nabla\mathbf{w} - \mu\sigma^2\mathbf{w} = \nabla P - \mathbf{e}\delta(\mathbf{x}-\mathbf{y}) \tag{2.4}$$

(derivatives are with respect to $\mathbf{x}$).

We want to prove that $\mathbf{u}^N$ converges (in a sense which will be made precise in the sequel) to the macroscopic vector field $\mathbf{v}(\mathbf{x})$ which is defined as the solution of

$$\begin{aligned} \mu\,\Delta\mathbf{v} - \lambda\mathbf{v} - 6\pi\mu\alpha'\rho(\mathbf{x})\mathbf{v} &= \nabla\bar{P} + \mathbf{f} \qquad \text{in} \quad D \\ \nabla\cdot\mathbf{v} &= 0 \qquad \text{in} \quad D \end{aligned} \tag{2.5}$$

(2.5) is sometimes referred to as Brinkman's equation (Saffman,[15] Lundgren,[11] and Tam[16]).

We start with a few preliminary propositions:

Lemma 1. Let Ω be the configurations ensemble of particle centers, and $\Omega_1^N \subseteq \Omega$ be that part of Ω for which

$$\min_{i\neq j}|\mathbf{y}_i - \mathbf{y}_j| \geqslant cN^{-1+\nu} \qquad \nu < \tfrac{1}{3} \tag{2.6}$$

$$\frac{1}{N^2}\sum_{i,j}{}'|\mathbf{y}_i - \mathbf{y}_j|^{-3+\xi} \leqslant c < \infty \quad \xi > 0 \tag{2.7}$$

and

$$\frac{1}{N^3}\sum_{\substack{i,j,k\\ i\neq j\neq k}} |\mathbf{y}_i - \mathbf{y}_j|^{-2}|\mathbf{y}_j - \mathbf{y}_k|^{-2} < c < \infty \tag{2.8}$$

(c will denote a generic constant, and $\sum_j' \equiv \sum_{j,j\neq i}$).

Then

$$\lim_{N\to\infty} P_\Omega(\Omega - \Omega_1^N) = 0$$

A proof is given by Ozawa.[12] (He treats only (2.6) and (2.7), but (2.8) can be included using the same method).

From now on we will assume $\{\mathbf{y}_j\}$ satisfy (2.6)–(2.8), and hence our result will hold in probability. Notice that (2.6) excludes the possibility of overlapping.

Next, let us define a $3N\times 3N$ matrix $\tilde{\psi}$ that will play a major role in the analysis. $\tilde{\psi}$ will be the $N\times N$ blocks matrix where the (i, j) block is $\psi(\mathbf{y}_i, \mathbf{y}_j)$ for $i\neq j$, and the 3×3 zero matrix for $i=j$. i, j: 1, 2,..., N.

Lemma 2.

$$\left\|\frac{1}{N}\tilde{\psi}\right\| \leqslant c\sigma^{-\beta} \qquad \forall\beta<\frac{1}{2}$$

$\|\cdot\|$ denotes matrix (or vectors) norm, $\sigma^2=\lambda/\mu$.

Proof. Rewriting $\psi(\mathbf{x}, 0)$

$$(\psi)_{ij}=\frac{e^{-\sigma|\mathbf{x}|}}{|\mathbf{x}|}\delta_{ij}+\frac{x_i x_j}{|\mathbf{x}|^3}e^{-\sigma|\mathbf{x}|}$$
$$+\frac{1-e^{-\sigma|\mathbf{x}|}-\sigma|x|\,e^{-\sigma|\mathbf{x}|}}{\sigma^2|\mathbf{x}|^2}\left(-\frac{\delta_{ij}}{|\mathbf{x}|}+\frac{3x_i x_j}{|\mathbf{x}|^3}\right)$$

and observing that

$$e^{-\sigma|\mathbf{x}|} \leqslant (\sigma|\mathbf{x}|)^{-\beta} \qquad \beta<\tfrac{1}{2}$$
$$\frac{1-e^{-\sigma|\mathbf{x}|}-\sigma|\mathbf{x}|\,e^{-\sigma|\mathbf{x}|}}{\sigma^2|\mathbf{x}|^2}\leqslant(\sigma|\mathbf{x}|)^{-\beta}$$

we get

$$\|\psi(\mathbf{y}_i, \mathbf{y}_j)\|^2\leqslant c\frac{\sigma^{-2\beta}}{|\mathbf{y}_i-\mathbf{y}_j|^{2+2\beta}}$$

So

$$\left\|\frac{1}{N}\tilde{\psi}\right\|\leqslant c\left(N^{-2}\sum_{i,j}{}'\frac{\sigma^{-2\beta}}{|\mathbf{y}_i-\mathbf{y}_j|^{2+2\beta}}\right)^{1/2}\leqslant c\sigma^{-\beta}$$

where we used (2.7). Q.E.D.

Unlike the diffusion equation, the Stokes equation does not satisfy the maximum principle. There are, however, certain a priori estimates and variational principles which hold for it. The following proposition will be used in the next section.

Lemma 3. Let A be a bounded domain in R^3

$$\mu \Delta \mathbf{z} - \lambda \mathbf{z} = \nabla P \qquad \text{in} \quad A$$

$$\nabla \cdot \mathbf{z} = 0 \qquad \text{in} \quad A$$

$$\mathbf{z}|_{\partial A} = \mathbf{U}$$

Set

$$F(\mathbf{u}) = \mu \int_A u_{i,j} u_{i,j} + \lambda \int_A u_i u_i$$

and let G be the class of piecewise differentiable divergence free vector fields in A, satisfying $\mathbf{u}|_{\partial A} = \mathbf{U}$.

Then

$$F(\mathbf{z}) = \min_{\mathbf{u} \in G} F(\mathbf{u})$$

Proof. Let $\mathbf{w} \in G$ and write $\mathbf{w} = \mathbf{z} + \mathbf{u}$.

$$F(\mathbf{w}) = F(\mathbf{z}) + F(\mathbf{u}) + 2\mu \int_A u_{i,j} z_{i,j} + 2\lambda \int_A u_i z_i$$

but $u_{i,j} z_{i,j} = \partial_j (u_i z_{i,j}) - u_i z_{i,jj}$, so

$$F(\mathbf{w}) = F(\mathbf{z}) + F(\mathbf{u}) + 2 \int_{\partial A} u_i z_{i,j} n_j \, ds - 2 \int u_i (\mu z_{i,jj} - \lambda z_i)$$

using $u_i|_{\partial A} = 0$ and $u_{i,i} = 0$ we conclude

$$F(\mathbf{w}) = F(\mathbf{z}) + F(\mathbf{u}) \geqslant F(\mathbf{z}) \qquad \text{Q.E.D.}$$

Note that for finite μ and λ, $F(\mathbf{u})$ is equivalent to the W_2^1 norm, where

$$\|g\|_{2,1}(A) = \int_A |\nabla g|^2 + |g|^2$$

The main result we are going to prove is:

Theorem 1. Let $\mathbf{u}^N, \mathbf{v}$ be the solution of (2.2), (2.5), respectively, under the conditions specified above. Then

$$\lim_{N \to \infty} P_\Omega\{N^\gamma \|\mathbf{u}^N - \mathbf{v}\|_2(D) < \varepsilon\} = 1 \qquad \forall \gamma < \tfrac{1}{6}, \qquad \varepsilon > 0, \quad \lambda \geqslant \lambda_0$$

$(\mathbf{u})^N$ is extended from D_1 to D by defining it to be identically zero in UB_j, and λ_0 is a constant depending on D and α' but not on N).

3. POINT-SOURCES APPROXIMATION

Let $\mathbf{w}^N\mathbf{e}$ be the Green's function associated with (2.2). (A source of strength $\mathbf{e}$ is located at $\mathbf{x} \in D_1^N$). The idea is to approximate $\mathbf{w}^N\mathbf{e}$ by point sources distributed at $\{\mathbf{y}_j\}$ with suitable strengths. For this purpose we let

$$\mathbf{z}_e^N = \mathbf{\Psi}(\mathbf{x}, \mathbf{y})\mathbf{e} + \sum_j \boldsymbol{\psi}(\mathbf{y}_j, \mathbf{y})\mathbf{q}_x^j, \qquad \mathbf{q}_x^j = (q_{1x}^j, q_{2x}^j, q_{3x}^j) \tag{3.1}$$

Where the $\mathbf{q}_x$ are to be found from an appropriate set of boundary conditions. Point sources do not suffice however, to account for the full boundary conditions in (2.2). We ask instead that the average value of $\mathbf{z}_e^N$ on each S_i be zero. Averaging $\boldsymbol{\psi}(\mathbf{x}, \mathbf{y})$ for $\mathbf{y} \in S_i$, neglecting the exponential factor (recall that N is large) and using

$$\left\langle \frac{\partial^2}{\partial x_i \, \partial x_j} |\mathbf{x}| \right\rangle = \frac{1}{3} \delta_{ij} \cdot \frac{1}{R}$$

where $\langle \cdot \rangle$ means surface average over a sphere of radius R, we arrive at the following system of equations for $\{\mathbf{q}_x^j\}$

$$\boldsymbol{\psi}(\mathbf{x}, \mathbf{y}_i)\mathbf{e} + \sum_j{}' \boldsymbol{\psi}(\mathbf{y}_j, \mathbf{y}_i)\mathbf{q}_x^j + \frac{N}{\alpha} I_3 \mathbf{q}_x^i = 0, \qquad i = 1, 2, ..., N \tag{3.2}$$

Here $\alpha = 6\pi\mu\alpha'$ and I_K is the $K \times K$ identity matrix.

Construct now a vector $\tilde{\mathbf{q}}$ of length $3N$, which is composed of N chains of length 3, the kth chain being $\mathbf{q}_x^k$, the matrix $\tilde{\boldsymbol{\psi}}$ as in Section 2, and the $3N \times 3$ matrix $\tilde{\mathbf{\Phi}}(\mathbf{x}, \{y_j\})$

$$\tilde{\mathbf{\Phi}} = \begin{matrix} \boldsymbol{\psi}(\mathbf{x}, \mathbf{y}_1) \\ \vdots \\ \boldsymbol{\psi}(\mathbf{x}, \mathbf{y}_N) \end{matrix}$$

Then, (3.2) can be written as

$$\tilde{\mathbf{\Phi}}\mathbf{e} + \left(\tilde{\boldsymbol{\psi}} + \frac{N}{\alpha} I_{3N} \right) \tilde{\mathbf{q}}_x = 0$$

so

$$\tilde{\mathbf{q}}_x = \left(\frac{\alpha}{N}\tilde{\psi} + I_{3N}\right)^{-1}\left(-\frac{\alpha}{N}\tilde{\mathbf{\Phi}}\mathbf{e}\right) \tag{3.3}$$

From Lemma 2, $\|(\alpha/N)\tilde{\mathbf{\Psi}}\| \leqslant c\sigma^{-\beta}$, so $\tilde{\mathbf{q}}_x$ is well-defined for λ large enough. Returning to (3.1) we rewrite $\mathbf{z}^N$ as

$$\mathbf{z}_e^N = \mathbf{\Psi}(\mathbf{x}, \mathbf{y})\mathbf{e} - \frac{\alpha}{N}\tilde{\mathbf{\Phi}}^T(\{\mathbf{y}_i\}, \mathbf{y})\left(\frac{\alpha}{N}\tilde{\psi} + I_{3N}\right)^{-1}\tilde{\mathbf{\Phi}}\mathbf{e} \tag{3.4}$$

Note that

$$\mathbf{u}^N(\mathbf{y}) = \int \mathbf{w}^N(\mathbf{x}, \mathbf{y})\,\mathbf{f}(\mathbf{x})\,d\mathbf{x} \tag{3.5}$$

and similarly we set

$$\mathbf{\eta}^N(\mathbf{y}) = \int \left[\psi(\mathbf{x}, \mathbf{y})\,\mathbf{f}(\mathbf{x})\,d\mathbf{x} - \frac{\alpha}{N}\tilde{\mathbf{\Phi}}^T(\mathbf{y})\left(\frac{\alpha}{N}\tilde{\psi} + I_{3N}\right)^{-1}\tilde{\mathbf{\Phi}}(\mathbf{x})\,\mathbf{f}(\mathbf{x})\right]d\mathbf{x} \tag{3.6}$$

Theorem 2.

$$\|\mathbf{\eta}^N(\mathbf{y}) - \mathbf{u}^N(\mathbf{y})\|_2\,(D_1^N) \leqslant cN^{-\gamma} \qquad \text{for any} \quad \gamma < \tfrac{1}{6}, \quad \lambda \geqslant \lambda_0$$

and $\{\mathbf{y}_j\} \in \Omega_1^N$.

Proof. Let $\mathbf{\delta}^N = \mathbf{\eta}^N(\mathbf{y}) - \mathbf{u}^N(\mathbf{y}) \cdot \mathbf{\delta}^N$ solves

$$\begin{aligned} \mu\,\Delta\mathbf{\delta}^N - \lambda\mathbf{\delta}^N &= \nabla\tilde{P}^N && \text{in} \quad D_1^N \\ \nabla\cdot\mathbf{\delta}^N &= 0 && \text{in} \quad D_1^N \\ \mathbf{\delta}^N(\mathbf{y}) &= \mathbf{\eta}^N(\mathbf{y}) && \text{for} \quad \mathbf{y}\in S_i \quad i = 1, 2,..., N \end{aligned} \tag{3.7}$$

We want to use the variational principle (Lemma 3). First we need an estimate for $\mathbf{\eta}^N(\mathbf{y})$ on the boundaries. Recalling (3.2) we write

$$\begin{aligned} \mathbf{\eta}^N(\mathbf{y}) = \int [\psi(\mathbf{x}, \mathbf{y}) - \psi(\mathbf{x}, \mathbf{y}_i)]\,\mathbf{f}(\mathbf{x})\,d\mathbf{x} + \sum_j{}' \{[\psi(\mathbf{y}_j, \mathbf{y}_i)] - \psi(\mathbf{y}_j\cdot \mathbf{y})\}\,\mathbf{q}_f^j \\ + \left[\psi(\mathbf{y}_i, \mathbf{y}) - \frac{N}{\alpha}I_3\right]\mathbf{q}_f^i \qquad \mathbf{y}\in S_i \end{aligned} \tag{3.8}$$

where

$$\tilde{\mathbf{q}}_f = -\frac{\alpha}{N}\int\left(\frac{\alpha}{N}\tilde{\mathbf{\Psi}} + I_{3N}\right)^{-1}\tilde{\mathbf{\Phi}}(\mathbf{x}, \{\mathbf{y}_i\})\,\mathbf{f}(\mathbf{x})\,d\mathbf{x} \tag{3.9}$$

From the potential estimate

$$\sup_x \left| \int \frac{f(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^\alpha}\, d\mathbf{y} \right| \leqslant \|f\|_p, \qquad p > \frac{1}{1-(\alpha/3)} \tag{3.10}$$

we get

$$|\boldsymbol{\eta}^N(\mathbf{y})| \leqslant \frac{c}{N}\|\mathbf{f}\|_{q_1} + \frac{c}{N}\sum_j{}' \|\mathbf{q}_f^j\|\, |\mathbf{y}_j-\mathbf{y}_i|^{-2} + c\|\mathbf{q}_f^i\|, \qquad \mathbf{y}\in S_i, \quad q_1>3 \tag{3.11}$$

Another useful estimate which is obtained from (3.9), (3.10), and Lemma 2 is

$$\|\tilde{\mathbf{q}}_f\| \leqslant cN^{-1/2}\|\mathbf{f}\|_2 \tag{3.12}$$

We turn now to construct a candidate field $\boldsymbol{\tau}$ to be used in the variational principle. Let $\{T_i\}$ $i=1, 2,..., N$ be the following domains

$$T_i = \{\mathbf{x};\, \alpha'/N \leqslant |\mathbf{x}-\mathbf{y}_i| \leqslant 2\alpha'/N\}$$

From (2.6) these are "security domains." In each T_i let $\boldsymbol{\tau}^i$ be the vector satisfying

$$\begin{aligned} &\nabla\cdot\boldsymbol{\tau}^i = 0 && \text{in} \quad T_i \\ &\boldsymbol{\tau}^i(\mathbf{y}) = \boldsymbol{\eta}^N(\mathbf{y}) && y\in S_i \\ &\boldsymbol{\tau}^i(\mathbf{y}) = 0 && |\mathbf{y}-\mathbf{y}^i| = 2\alpha'/N \end{aligned} \tag{3.13}$$

Such a vector exists since the compatiblity condition ($\int_{S_i} \boldsymbol{\eta}^N\cdot\mathbf{n}\, ds = 0$) is satisfied by the definition of $\boldsymbol{\eta}^N$.

For smooth domains, there are a priori estimates for problems like (3.13). In order to use them we rescale

$$\mathbf{x} = \mathbf{z}/N$$

and observe that

$$\|\boldsymbol{\tau}^i(\mathbf{x})\|_{2,1} \leqslant N^{-1/2}\|\boldsymbol{\tau}^i(\mathbf{z})\|_{2,1} \tag{3.14}$$

But on the z scale, problem (3.13) is considered in a smooth domain, and we can use the estimate

$$\|\boldsymbol{\tau}^i\|_{2,1}(T_i) \leqslant c\,\|\boldsymbol{\eta}^N\|_{2,1/2}(\partial T_i) \qquad \text{(Ladyzenskaya, Ref. 10)}$$

$\int_{\partial T_i} ds$ and $\int_{\partial T_i}\int_{\partial T_i} (dz\,dz'/|\mathbf{z}-\mathbf{z}'|^3)$ are bounded, hence it follows from (3.11) that

$$\|\boldsymbol{\eta}^N\|_{2,1/2}(\partial T_i) \leqslant \frac{c}{N}\|\mathbf{f}\|_{q_1} + \frac{c}{N}\sum_j{}' \|\mathbf{q}_f^j\|\,|\mathbf{y}_j-\mathbf{y}_i|^{-2} + c\,\|\mathbf{q}_f^i\| \qquad (3.15)$$

We now define $\boldsymbol{\tau}$ to be $\boldsymbol{\tau}^i$ in T_i and zero elsewhere.

$$\|\boldsymbol{\tau}\|_{2,1}(D_1^N) = \sum_i \|\boldsymbol{\tau}^i(x)\|_{2,1}(T_i)$$
$$\leqslant N^{-1/2}\sum_i \frac{c}{N}\|\mathbf{f}\|_{q_1} + c\cdot N^{-3/2}\sum_{i,j}{}' \|\mathbf{q}_f^j\|\,|\mathbf{y}_j-\mathbf{y}_i|^{-2} + cN^{-1/2}\sum_i \|\mathbf{q}_f^i\|$$

where we use (3.14) and (3.15).

Let now $\mathbf{Q}$ be the matrix

$$(\mathbf{Q})_{ij} = \begin{matrix} |\mathbf{y}_j-\mathbf{y}_i|^{-2} & i\neq j \\ 0 & i=j \end{matrix}$$

$$\|\mathbf{Q}\| = \left(\sum_{i,j}{}' |\mathbf{y}_i-\mathbf{y}_j|^{-4}\right)^{1/2} \leqslant cN^{1+[(1-\nu)(1+\xi)/2]} \qquad (3.16)$$

for $\nu<\frac{1}{3}$, $\xi>0$, and (2.6) and (2.7).

Hence (applying Cauchy Schwatz inequality)

$$\|\boldsymbol{\tau}\|_{2,1}(D^N) \leqslant cN^{-1/2}\|\mathbf{f}\|_{q_1} + cN^{-3/2}\|\tilde{\mathbf{q}}_f\|\,\|\mathbf{Q}\|\cdot N^{1/2} + c\,\|\mathbf{q}_f\|$$
$$\leqslant cN^{-1/2}\|\mathbf{f}\|_{q_1} + cN^{-(1/6)+\varepsilon}\|\mathbf{f}\|_2$$

(In the last step we used (3.12) and (3.16)).

Recalling Lemma 3 and the remark on the equivalence of $\|\cdot\|_{2,1}$ and $F(\cdot)$, we obtain

$$\|\boldsymbol{\delta}^N\|_{2,1} \leqslant cN^{-\gamma} \qquad \gamma<\tfrac{1}{6}$$

which is in fact stronger than the statement of the theorem. Q.E.D.

Remarks.

1. It is obvious from the proof that we could have used weaker conditions on $\mathbf{f}$. For example, we can take $\mathbf{f}\in L_q(D)$, $q>3$ with fast decay for large $|\mathbf{x}|$.
2. $\boldsymbol{\eta}^N(\mathbf{y})$ can be extended to all D, and the difference between the full vector field and the reduced one can be shown to be less than $cN^{-1/2}$ (in L_2 norm). The proof is exactly the same as the one given by Figari, Orlandi, and Teta.[5] Notice also that in our scaling

$$\mathrm{Vol}(D-D_1^N) = O(N^{-2}) \to 0 \qquad \text{as} \quad N\to\infty$$

4. HOMOGENIZATION OF THE POINT-SOURCES APPROXIMATION

The work we have put into the point-sources approximation pays off now, since we do not have to deal anymore with the complicated geometry. Our next step is to prove that $\boldsymbol{\eta}^N$ converges in the appropriate sense to $\mathbf{v}$. We follow here the homogenization procedure suggested by Figari, Orlandi, and Teta.[5] Even though their work dealt with the Laplace equation, a careful examination of their homogenization theorem shows that it applies to a very general family of equations, among them the Stokes equation. The idea is to compare the resolvent expansion of $\boldsymbol{\eta}^N(\mathbf{y})$ and $\mathbf{v}(\mathbf{y})$. For every $\boldsymbol{\Gamma}$ we let

$$\boldsymbol{\Gamma}_f \equiv \int \boldsymbol{\Gamma}(\mathbf{x}, \mathbf{y})\, \mathbf{f}(\mathbf{x})\, d\mathbf{x}, \qquad {}_f\boldsymbol{\Gamma} \equiv (\boldsymbol{\Gamma}_f)^{\mathrm{T}} \tag{4.1}$$

Then

$$\mathbf{v}(\mathbf{y}) = \sum_{n=0}^{\infty} \boldsymbol{\psi} \cdot (-\alpha)^n (\boldsymbol{\psi}\rho)_f^n \tag{4.2}$$

or

$$\begin{aligned} v_\beta(\mathbf{y}) = \sum_{n=0}^{\infty} (-\alpha)^n \int & \boldsymbol{\psi}_{\beta\beta_1}(\mathbf{y}, \mathbf{z}_1)\, \rho(\mathbf{z}_1) \\ & \times \boldsymbol{\psi}_{\beta_1\beta_2}(\mathbf{z}_1, \mathbf{z}_2)\, \rho(\mathbf{z}_2) \cdots \boldsymbol{\psi}_{\beta_{n-1}\beta_n}(\mathbf{z}_{n-1}, \mathbf{z}_n)\, \rho(\mathbf{z}_n) \\ & \times \boldsymbol{\psi}_{\beta_n\beta_{n+1}}(\mathbf{z}_n, \mathbf{z}_{n+1})\, \mathbf{f}_{\beta_{n+1}}(\mathbf{z}_{n+1}) \cdot d\mathbf{z}_1 \cdots d\mathbf{z}_{n+1} \end{aligned} \tag{4.3}$$

where $\beta_i = 1, 2, 3$. Also

$$\boldsymbol{\eta}^N(y) = \boldsymbol{\psi}_f + \sum_{n=1}^{\infty} \left(-\frac{\alpha}{N}\right) \tilde{\boldsymbol{\Phi}} \left(-\frac{\alpha}{N}\right)^{n-1} (\tilde{\boldsymbol{\psi}})^{n-1}\, \tilde{\boldsymbol{\Phi}}_f \tag{4.4}$$

A general term in this expansion is of the form

$$(\alpha/N)^s\, \boldsymbol{\psi}_{\beta\beta_1}(\mathbf{y}, \mathbf{y}_{j_i})\, \boldsymbol{\psi}_{\beta_1\beta_2}(\mathbf{y}_{j_1}, \mathbf{y}_{j_2}) \cdots \boldsymbol{\psi}_{\beta_{s-1}\beta_s}(\mathbf{y}_{j_{s-1}}, \mathbf{y}_{j_s})(\boldsymbol{\psi}_{\mathrm{f}})_{\beta_s}(\mathbf{y}_{j_s}) \tag{4.5}$$

We would like to dispose of terms which contain repeating points $\mathbf{y}_k$. (This will make our averaging procedure much easier.)

We decompose $\boldsymbol{\eta}^N$ into

$$\boldsymbol{\eta}^N(\mathbf{y}) = \mathbf{v}^N(\mathbf{y}) + \zeta^N(\mathbf{y}) \tag{4.6}$$

where $\zeta^N(y)$ contains all the terms with "self-intersecting graphs," and $\mathbf{v}^N$ contains the rest, i.e., all the terms in (4.4) in which each point appears at most once.

The following Lemma holds (see also Remark 5 in Sect. 5):

Lemma 4. There is λ_0, such that for every $\lambda \geqslant \lambda_0$ and for every $\varepsilon > 0$ there is N_0, such that for every $N \geqslant N_0$ and for $\{\mathbf{y}_j\} \in \Omega_1^N$

$$\|N^{1/2}\zeta^N\|_2 \leqslant \varepsilon \, \|\mathbf{f}\|_2 \qquad \forall \mathbf{f} \in C_0(D)$$

The proof is given in the Appendix.

In the sequel, we emphasize the dependence of $\boldsymbol{\eta}^N$ of $\mathbf{f}$ by writing it as $\boldsymbol{\eta}^N(\mathbf{f})$. Let us define for every $\mathbf{f} \in L_2(D)$ and $\mathbf{g}$ which is continuous and has a compact support in D the random variable

$$\theta^N(\mathbf{f}, \mathbf{g}) = \sum_{s=0}^{N} (-\alpha)^s \cdot \{N^{-s+(1/2)}{}_f\boldsymbol{\Phi}(\tilde{\boldsymbol{\psi}}^{s-1})_* \, \tilde{\boldsymbol{\Phi}}_g - N^{1/2}[\mathbf{f}, \underset{\sim}{\boldsymbol{\psi}}(\boldsymbol{\psi}\rho)_g^s]\} \tag{4.7}$$

where $(\mathbf{u}, \mathbf{v})$ is the scalar product in $L_2(D)$ of vector fields and $(\tilde{\boldsymbol{\psi}}^n)_*$ means that we count only the terms which belong to $\mathbf{v}^N(\mathbf{g})$.

Using the resolvent expansions and the fact that $(\tilde{\boldsymbol{\psi}}^n)_*$ does not contain repeating points we see that

$$\lim_{N \to \infty} \theta^N(\mathbf{f}, \mathbf{g}) - N^{1/2}[\mathbf{f}, \mathbf{v}^N(\mathbf{g}) - \mathbf{v}(\mathbf{g})] = 0 \tag{4.8}$$

Taking expection on $\theta^N(\mathbf{f}, \mathbf{g})$ means taking expectations on terms like (4.5), only that now the $\mathbf{y}_j$ are all distinct. Hence, the expectation of such a term is precisely the corresponding term in the resolvent expansion of $\mathbf{v}_\beta$ (4.3). We only have to count how many such terms (like (4.5) with distinct $\mathbf{y}_j$) appear, and one finds

$$E[\theta^N(\mathbf{f}, \mathbf{g})] = N^{1/2} \sum_{s=1}^{N} [\gamma(N, s) - 1](-\alpha)^s [\mathbf{f}, \underset{\sim}{\boldsymbol{\psi}}(\boldsymbol{\psi}\rho)_g^s] \tag{4.9}$$

where $\gamma(N, s) = N!/(N-s)! \, N^s$.

Lemma 5.

$$\lim_{N \to \infty} E[\theta^N(\mathbf{f}, \mathbf{g})] = 0$$

Proof. Notice first (from the proof of Lemma 2, for example) that

$$|(\mathbf{f}, \boldsymbol{\psi}(\boldsymbol{\psi}\rho)_g^s)| \leqslant c\lambda^{-s}$$

Consider now s: 1, 2,..., lg N for N large. Then $\gamma(N, s)$ can be expanded as

$$\gamma(N, s) \simeq 1 - \frac{s^2}{2N} + \cdots$$

and for any s, $\gamma(N, s) \leqslant 1$.

Combining these observations we estimate

$$|E[\theta^N(\mathbf{f},\mathbf{g})]| \leqslant \frac{\lg^2 N}{N^{1/2}} \sum_{s=1}^{\lg N} (c\lambda)^{-s} + cN^{1/2} \sum_{s=\lg N+1}^{\infty} (c\lambda)^{-s} \to 0$$

as $N \to \infty$ for λ large enough. Q.E.D.

Next we compute the covariance for $\mathbf{f}, \mathbf{f}'$

$$\begin{aligned} &E[\theta^N(\mathbf{f},\mathbf{g})\,\theta^N(\mathbf{f}',\mathbf{g})] \\ &\quad = \sum_{s,s'=1}^{N} (-\alpha)^{s+s'} E[N^{-s-s'+1}\,{}_f\tilde{\mathbf{\Phi}}(\tilde{\psi})_*^{s-1}\,\tilde{\mathbf{\Phi}}_{gf'}\,\tilde{\mathbf{\Phi}}(\tilde{\psi})_*^{s'-1}\tilde{\mathbf{\Phi}}_g] \\ &\qquad + N[1-\gamma(N,s)-\gamma(N,s')][\mathbf{f},\psi(\psi\rho)_g^s][\mathbf{f}',\psi(\psi\rho)_g^{s'}] \end{aligned} \tag{4.10}$$

A tedius computation, which is basically similar to our proof of Lemma 5, yields

Lemma 6.

$$E[\theta^N(\mathbf{f},\mathbf{g})\,\theta^N(\mathbf{f}',\mathbf{g})] = G(\mathbf{f},\mathbf{f}',\mathbf{g})$$

where G is a bounded functional on $L_2(D) \times L_2(D) \times L_2(D)$. (An explicit formula for G can be given, but we do not find it very illuminating, especially when Theorem 2 gives a convergence rate which is less than $\frac{1}{6}$).

Finally, we extend $\mathbf{u}^N(\mathbf{y})$ into D by defining it to be identically zero in $\bigcup_{j=1}^N B_j$. Lemma 6, Lemma 5, Lemma 4, (4.8), Remark 3 in Section 3, Theorem 2, and Lemma 1 altogether complete the proof of Theorem 1.

5. SOME REMARKS

1. Our proof is valid for $\lambda \geqslant \lambda_0$, where λ_0 depends on α' and the domain. Had we worked with the time variable we would get Theorem 1 for $t \leqslant T < \infty$. Implicit in our formulation is the assumption that (2.2) describes the microscopic flow. This assumption means that we are actually studying the decay of a momentum perturbation for a flow with small Reynolds number in random media.
2. Brinkman's equation is often written in the form

$$\mu' \Delta \mathbf{v} - \frac{\mu}{K}\mathbf{v} = \nabla P \tag{5.1}$$

where K is the permeability of the porous medium and μ' is some effective viscosity (Lundgren[11]). It follows from our derivation that

$$\mu' = \mu, \qquad K = \frac{1}{6\pi\alpha'\rho}$$

or

$$K = \frac{2}{9} R^2 \cdot \beta^{-1} \tag{5.2}$$

where β is the volume fraction occupied by the spheres. Equation (5.2) is the expression for K that was derived by Brinkman for dilute suspensions.

3. The scaling we used here corresponds to very low volume fraction of obstacle (high porosity). What happens in higher volume fractions? This is still an open question, although there have been some attempts to find corrections to (2.5). (See, for example, Childress[4] and Hinch.[17]) We believe that a possible approach is to improve Theorem 2 by going to higher multipoles.
4. It is not hard to check that the proof holds for a distribution of non-identical spheres with $R_j = \alpha'_j/N$, provided that $0 < \alpha_- \leqslant \alpha'_j \leqslant \alpha_+ < \infty$, and that there exists a function $\bar{\alpha}'(\mathbf{x})$ such that

$$\frac{1}{N}\sum_{j=1}^{N} \alpha'_j \phi(\mathbf{y}_j) \to \int \rho(\mathbf{y})\, \bar{\alpha}'(\mathbf{y})\, \phi(\mathbf{y})\, d\mathbf{y} \qquad \text{as} \quad N \to \infty$$

for every continuous ϕ.
5. It should be noted that Lemma 4 is a crucial step in the proof. It is very important to control the terms in the expansion (4.4) for two reasons:
   (i) Graphs which contain too many loops do not have moments (with respect to P_Ω).
   (ii) The terms with nonintersecting graph are much easier to handle since we can average them "layer by layer."

APPENDIX

The proof of Lemma 4 is similar to the one given in Ref. 5 but we give it here explicitly because of its importance.

We notice first that for $\mathbf{f} \in C_0(D)$, $\boldsymbol{\eta}^N(\mathbf{f}) \in L_2(D)$. Consider now (for $\sigma > 0$)

$$N^{1-\sigma}[\mathbf{h}, \zeta^N(\mathbf{f})] \qquad \text{for} \quad \mathbf{h} \in L_2, \quad \mathbf{f} \in C_0(D)$$

Let the first point which repeats itself (see (4.4), (4.5)) be at the n_1 place in the expansion, and assume that it appears again at the n_1+n_2 place. The contribution of such a term is less than

$$T_{n_1 n_2} = \frac{\alpha^{n+1}}{N^{n+\sigma}} \sum_{\substack{i,j,p,k \\ r,q,l=1}}^{3N} ({}_h\boldsymbol{\Phi})_i (\tilde{\boldsymbol{\psi}}^{n_1-1})_{ip} (\tilde{\boldsymbol{\psi}})_{pk} (\tilde{\boldsymbol{\psi}})_{kl} (\tilde{\boldsymbol{\psi}}^{n_2-1})_{lq} (\tilde{\boldsymbol{\psi}})_{qk} (\tilde{\boldsymbol{\psi}})_{k'r} \cdot (\tilde{\boldsymbol{\psi}}^{n-n_1-n_2-1})_{rj} (\boldsymbol{\Phi}_f)_j$$

where $|k-k'|<3$.

Using the potential estimate (3.8) (recall that $(\tilde{\boldsymbol{\psi}})_{ij} \leqslant (c/|\mathbf{y}_i - \mathbf{y}_j|)$ and applying Cauchy–Schwartz inequality repeatedly we find

$$T_{n_1 n_2} \leqslant c\alpha^{n+1} \|\mathbf{h}\|_2 \, \|\mathbf{f}\|_2 \left\| \frac{1}{N} \tilde{\boldsymbol{\psi}} \right\|^{n-4} \cdot N^{-\sigma} \cdot L(N)$$

where

$$L(N) \leqslant N^{-3} \sum_{\substack{i,j,k \\ i \neq j \neq k}} |\mathbf{y}_i - \mathbf{y}_j|^{-2} \, |\mathbf{y}_j - \mathbf{y}_k|^{-2} + N^{-3} \sum_{i,j}{}' \, |\mathbf{y}_i - \mathbf{y}_j|^{-4}$$

From (2.7) and (2.8), $L(N) \leqslant c$ and for every n there are at most n^2 terms like $T_{n_1 n_2}$. Recalling Lemma 2 we arrive at

$$N^{1-\sigma}[h, \zeta^N(\mathbf{f})] \leqslant \|\mathbf{f}\|_2 \, \|\mathbf{h}\|_2 \, N^{-\sigma} \sum_{n=2}^{\infty} (c\lambda)^{-n\beta/2} n^2$$

where the sum on the right converges for λ big enough. Q.E.D.

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REFERENCES

1. R. A. Adams, *Sobolev Spaces* (Academic Press, New York, 1975).
2. H. C. Brinkman, *Appl. Sci. Res. A* **1**:27 (1947).
3. R. Caflish and J. Rubinstein, Lectures on the Mathematical Theory of Multiphase Flow (Courant Institute of Mathematical Sciences, 1986) (1986).
4. S. Childress, *J. Chem. Phys.* **36**:2527 (1972).
5. R. Figari, E. Orlandi, and S. Teta, *J. Stat. Phys.* **41**:465 (1985).
6. D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order* (Springer-Verlag, New York, 1983).

7. J. Happel and H. Brenner, *Low Reynolds Number Hydrodynamics* (Prentice-Hall, Jersey, 1965).
8. E. Hruslov and V. A. Marchenko, *Boundary Value Problems in Regions with Fine Grained Boundaries* (Naurova Dunka, Kiev, 1974).
9. M. Kac, *Rocky Mn. J. Math.* **4**:511 (1974).
10. O. A. Ladyzhenzkaya, *The Mathematical Theory of Viscous Incompressible Flow* (Gordon and Breach, 1969).
11. T. S. Lundgren, *J.F.M.* **51**:273 (1972).
12. S. Ozawa, *Comm. Math. Phys.* **91**:473 (1983).
13. G. C. Papanicolaou and S. R. S. Varadhan, *Diffusion in Region with Many Small Holes*, in Lecture Notes in Control and Information, Vol. 75 (Springer, New York, 1980), pp. 190–206.
14. J. Rauch and M. Tatlor, *J. Funct. Anal.* **18**:27 (1975).
15. P. G. Saffman, *Stud. Appl. Math.* **50**:93 (1971).
16. C. K. W. Tam, *J.F.M.* **38**:537 (1969).
17. E. J. Hinch, *J.F.M.* **83**:695 (1977).